2016

BOOKLET NO. TEST CODE : **PSA**

Forenoon

Questions : 30	**Time : 2 hours**

Write your Registration Number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answersheet.

For each question, there are four suggested answers of which only one is correct. For each question indicate your choice of the correct answer by darkening the appropriate oval (●) completely on the answer sheet.

4 marks are allotted for each correct answer,
0 mark for each incorrect answer and
1 mark for each unattempted question.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET ONLY.
YOU ARE NOT ALLOWED TO USE CALCULATORS IN ANY FORM.

STOP! WAIT FOR THE SIGNAL TO START.

PSA$_e$

ROUGH WORK

1. The number of terms independent of x in the binomial expansion of $\left(3x^2 + \frac{1}{x}\right)^5$ is

(A) 1 (B) 5 (C) 0 (D) 2

2. Let X and Y be independent and identically distributed random variables with moment generating function

$$M(t) = \mathrm{E}\left(e^{tX}\right), \quad -\infty < t < \infty.$$

Then $\mathrm{E}\left(\frac{e^{tX}}{e^{tY}}\right)$ equals

(A) 1 (B) $\frac{M(t)}{M(-t)}$ (C) $(M(t))^2$ (D) $M(t)M(-t)$

3. A 6-digit number is to be formed by rearranging the digits of 654321. How many such numbers will be divisible by 12?

(A) 192 (B) 360 (C) 144 (D) 168

4. The value of

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{y} e^{-\frac{1}{2}(x^2+y^2)} \,\mathrm{d}x \,\mathrm{d}y$$

equals

(A) $\frac{1}{2\pi}$ (B) $\frac{\pi}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

5. Let A be a 2×2 matrix with real entries. If $5 + 3\sqrt{-1}$ is an eigenvalue of A, then the determinant of A equals

(A) 8 (B) 34 (C) 4 (D) 16

ROUGH WORK

6. If $A_1, \ldots, A_n$ are independent events with respective probabilities $p_1, \ldots, p_n$, then $\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right)$ equals

(A) $\sum_{i=1}^{n} p_i$ (B) $1 - \prod_{i=1}^{n}(1 - p_i)$ (C) $\prod_{i=1}^{n} p_i$ (D) $\prod_{i=1}^{n}(1 - p_i)$

7. Let A and B be matrices such that $B^2 + AB + 2I = \mathbf{0}$, where I denotes the identity matrix. Which of the following matrices must be nonsingular?

(A) $A + 2I$ (B) B (C) $B + 2I$ (D) A

8. Let X_1, X_2, and X_3 be independent Poisson random variables with mean 1. Then $\mathrm{P}(\max\{X_1, X_2, X_3\} = 1)$ equals

(A) $7e^{-3}$ (B) e^{-3} (C) $1 - e^{-3}$ (D) $1 - 8e^{-3}$

9. The number of solutions of the equation $2x^3 - 6x + 1 = 0$ in the interval $[-1, 1]$ is

(A) 2 (B) 3 (C) 1 (D) 0

10. Consider a confounded 2^5 factorial design with factors A, B, C, D, E arranged in four blocks each of size eight. If the principal block of this design consists of the treatment combinations $(1), ab, de, ace$, and four others, then the confounded factorial effects would be

(A) $AB, DE, ABDE$
(B) $ABC, CDE, ABDE$
(C) $ABC, DE, ABCDE$
(D) $AB, CDE, ABCDE$

ROUGH WORK

11. The number of ordered pairs (a, b) such that $a + b \leq 60$, where a and b are positive integers, is

(A) 3540 (B) 1770 (C) 1830 (D) 885

12. A box contains 2016 balls labeled $1, 2, 3, \ldots, 2016$. Two balls are selected at random by sampling without replacement. Let X_1 and X_2 be the labels on the first ball and the second ball, respectively. Then

(A) X_1 and X_2 are independent.

(B) $\mathrm{P}(X_1 < X_2) > \frac{1}{2}$.

(C) $\mathrm{E}(X_1) \neq \mathrm{E}(X_2)$.

(D) $\mathrm{E}(X_2 | X_1 = 1008) > 1008$.

13. An 8-digit number is to be formed with digits from the set $\{1, 2, 3\}$ such that the sum of the digits in the number is equal to 10. How many such numbers are there?

(A) $\binom{8}{1} + \binom{8}{2}$ (B) $\binom{8}{1} \times \binom{8}{2}$ (C) $\binom{8}{2}$ (D) $\binom{8}{1}$

14. Consider a finite population of size $N > 1$ with units $U_1, U_2, \ldots, U_N$. The following sampling method is used to select a sample: either the sample consists of only one unit U_j with probability $\frac{1}{N+1}$ for any $j = 1, 2, \ldots, N$, or it consists of the whole population with probability $\frac{1}{N+1}$. Then the expected sample size is

(A) 2 (B) $\frac{N+2}{N+1}$ (C) $\frac{2N}{N+1}$ (D) $\frac{2N+1}{N+1}$

15. For any θ, the expression $4 \sin\theta \sin(\frac{\pi}{3} + \theta) \sin(\frac{\pi}{3} - \theta)$ equals

(A) $\cos 3\theta$ (B) $\sin 3\theta$ (C) $\cos 2\theta$ (D) $\sin 2\theta$

ROUGH WORK

16. Let X be a Bernoulli $\left(\frac{1}{3}\right)$ random variable and Y be a Bernoulli $\left(\frac{2}{3}\right)$ random variable independent of X. Let

$$Z = \begin{cases} X & \text{if } Y = 1, \\ 1 - X & \text{if } Y = 0. \end{cases}$$

Given that $Z = 1$, the conditional probability that $X = 1$ is

(A) $\frac{2}{3}$ (B) $\frac{2}{9}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$

17. Let f be a polynomial such that $f''(x) \to 2$ as $x \to \infty$, the minimum of f is attained at 3, and $f(0) = 3$. Then $f(1)$ equals

(A) -2 (B) 1 (C) 2 (D) -1

18. Let X be a random variable with probability density function

$$f(x;\theta) = \begin{cases} \frac{\theta}{x}\left(\frac{4}{x}\right)^{\theta} & \text{if } x > 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$. If a test of size $\alpha = 0.1$ for testing $H_0 : \theta = 1$ vs $H_1 : \theta = 2$ rejects H_0 when $X < m$, then the value of m is

(A) $\frac{40}{3}$ (B) 5 (C) $\frac{80}{9}$ (D) $\frac{40}{9}$

19. Let g be a differentiable function from the set of real numbers to itself. If $g(1) = 1$ and $g'(x^2) = x^3$ for all $x > 0$, then $g(4)$ equals

(A) $\frac{32}{5}$ (B) $\frac{64}{5}$ (C) $\frac{37}{5}$ (D) $\frac{67}{5}$

ROUGH WORK

20. Suppose X and Z are two independent random variables such that $\mathrm{E}(X) = \mu > 0, \mathrm{Var}(X) = \sigma^2 > 0$, and $\mathrm{P}(Z = 1) = \mathrm{P}(Z = -1) = \frac{1}{2}$. Let $Y = ZX$. Then

(A) $\mathrm{Var}(Y) = \mu^2 + \sigma^2$.

(B) $\mathrm{E}(Y) > \mu$.

(C) X and Y have the same distribution.

(D) X and Y are always independent.

21. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many functions $f \colon A \to A$ are there such that $f(1) < f(2) < f(3)$?

(A) $\binom{8}{3}5^8$ (B) $\binom{8}{3}8^5$ (C) $\binom{8}{3}$ (D) $8!$

22. Let X and Y be independent Bernoulli random variables with

$$\mathrm{P}(X = 1) = p, \quad \mathrm{P}(Y = 1) = 1 - p.$$

Then the distribution of $X + Y - XY$ is

(A) Binomial $(2, \frac{1}{2}(1 - p + p^2))$

(B) Binomial $(2, p^2 + (1 - p)^2)$

(C) Bernoulli $(2p - 2p^2)$

(D) Bernoulli $(1 - p + p^2)$

23. Let $(a_i, \frac{1}{a_i}), i = 1, 2, 3, 4$, be four distinct points on the circle of radius 2 centred at the origin. Then the value of $a_1a_2a_3a_4$ is

(A) $\frac{1}{16}$ (B) 4 (C) -1 (D) 1

ROUGH WORK

24. Let X have a normal distribution with mean 0 and variance 2, and Y have a Poisson distribution with mean 1. Let

$$U = X + Y\,, \quad V = X - Y\,.$$

Then U and V are

(A) positively correlated.

(B) uncorrelated but not independent.

(C) negatively correlated.

(D) independent.

25. Let $f : (0, \infty) \to (0, \infty)$ be a strictly decreasing function. Consider

$$h(x) = \frac{f\left(\frac{x}{1+x}\right)}{1 + f\left(\frac{x}{1+x}\right)}\,, \quad x > 0.$$

Then

(A) h is strictly increasing.

(B) h is strictly decreasing.

(C) h has a minimum.

(D) h has a maximum.

26. Two integers m and n are chosen at random with replacement from $\{1, 2, \ldots, 9\}$. The probability that $m^2 - n^2$ is even is

(A) $\frac{37}{81}$ (B) $\frac{4}{9}$ (C) $\frac{2}{3}$ (D) $\frac{41}{81}$

27. Let f be a function from the set of real numbers to itself. If f is strictly increasing, then

(A) left and right limits, i.e., $\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$, exist for all a.

(B) $f'(a) > 0$ if f is differentiable at a.

(C) f is not bounded above.

(D) f is not bounded below.

ROUGH WORK

28. Suppose X_1 and X_2 are independent exponential random variables with means λ_1 and λ_2, respectively. To test

$$H_0 : \lambda_1 = \lambda_2 \text{ against } H_1 : \lambda_1 > \lambda_2,$$

consider a test which rejects H_0 if $X_1 > X_2$. What is the power of this test when $\lambda_1 = 2\lambda_2$?

(A) $\frac{8}{27}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{9}$

29. Let A be a 3×3 matrix with all diagonal elements equal to a real number x and all remaining entries equal to 1. The set of all possible values of the rank of A is

(A) $\{2, 3\}$ (B) $\{1, 3\}$ (C) $\{3\}$ (D) $\{1, 2, 3\}$

30. Given observed values $(y_1, x_1), (y_2, x_2), \dots, (y_n, x_n)$, the sum of squares

$$\sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$$

is minimised with respect to α and β to obtain estimators $\hat{\alpha}$ and $\hat{\beta}$. Suppose that the actual model is

$$\mathrm{E}(Y_i | X_i = x_i) = \alpha_0 + \frac{\beta_0}{c} x_i \quad \text{for all } i,$$

where $c \in (0, 1)$ is a fixed number and $\alpha_0 > 0$ and $\beta_0 > 0$ are unknown parameters. Then

(A) $\mathrm{E}(\hat{\alpha}) = \alpha_0$ and $\mathrm{E}(\hat{\beta}) > \beta_0$.

(B) $\mathrm{E}(\hat{\alpha}) \neq \alpha_0$ and $\mathrm{E}(\hat{\beta}) > \beta_0$.

(C) $\mathrm{E}(\hat{\alpha}) = \alpha_0$ and $\mathrm{E}(\hat{\beta}) = \beta_0$.

(D) $\mathrm{E}(\hat{\alpha}) \neq \alpha_0$ and $\mathrm{E}(\hat{\beta}) = \beta_0$.