

2014

BOOKLET No.

TEST CODE: PSB

Afternoon

Questions: 9	Time: 2 hours
---------------------	----------------------

Write your Registration number, Test Code, Number of this booklet, etc. in the appropriate places on the answer-booklet.

- All questions carry equal weight.
- Answer any two questions from GROUP A and any three questions from GROUP B.

Answer to each question should start on a fresh page.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET

AND/OR THE ANSWER-BOOKLET. YOU ARE

NOT ALLOWED TO USE CALCULATORS.

STOP! WAIT FOR THE SIGNAL TO START.

GROUP A

1. Let $f : R \rightarrow R$ be a function which is continuous at 0 and $f(0) = 1$.

Also assume that f satisfies the following relation for all x :

$$f(x) - f(x/2) = \frac{3x^2}{4} + x.$$

Find $f(3)$.

2. For any $n \times n$ matrix $A = ((a_{ij}))$, consider the following three properties:

1. a_{ij} is real valued for all i, j and A is upper triangular.
2. $\sum_{j=1}^n a_{ij} = 0$, for all $1 \leq i \leq n$.
3. $\sum_{i=1}^n a_{ij} = 0$, for all $1 \leq j \leq n$.

Define the following set of matrices:

$$\mathcal{C}_n = \{A : A \text{ is } n \times n \text{ and satisfies (1), (2) and (3) above}\}.$$

- (a) Show that $\mathcal{C}_n$ is a vector space for any $n \geq 1$.
 - (b) Find the dimension of $\mathcal{C}_n$, when $n = 2$ and $n = 3$.
3. Let A be a real valued and symmetric $n \times n$ matrix with entries such that $A \neq \pm I$ and $A^2 = I$.
- (a) Prove that there exist non-zero column vectors v and w such that $Av = v$ and $Aw = -w$.
 - (b) Prove that every vector z has a unique decomposition $z = x + y$ where $Ax = x$ and $Ay = -y$.

GROUP B

4. Suppose that 15 identical balls are placed in 3 boxes labeled A, B and C. What is the number of ways in which Box A can have more balls than Box C?
5. Suppose that X and Y are random variables such that

$$E(X + Y) = E(X - Y) = 0,$$

$$\text{Var}(X + Y) = 3,$$

$$\text{Var}(X - Y) = 1.$$

- (a) Evaluate $\text{Cov}(X, Y)$.
- (b) Show that $E(|X + Y|) \leq \sqrt{3}$.
- (c) If in addition, it is given that (X, Y) has a bivariate normal distribution, calculate $E(|X + Y|^k)$ for all positive integers k .
6. Suppose that $X_1, \dots, X_n$ are i.i.d. random variables with mean μ and variance 1. Also assume that $Y_1, \dots, Y_n$ are i.i.d. with probability mass function $P(Y_i = \pm 1) = 1/2$ for all $1 \leq i \leq n$ and independent of $X_1, \dots, X_n$. Define T_n as follows:

$$T_n = \frac{1}{n} \sum_{i=1}^n Y_i \cdot |X_i|, \quad n \geq 1.$$

- (a) For any fixed $z \in R$, find

$$\lim_{n \rightarrow \infty} P(\sqrt{n}T_n \leq z).$$

- (b) Using the result in part (a) above, find random quantities L_n and U_n , based on T_n , such that

$$\lim_{n \rightarrow \infty} P(L_n \leq \mu \leq U_n) = 0.95.$$

7. Suppose that $X_1, \dots, X_n$ are i.i.d. with probability density function

$$f_{\theta}(x) = \begin{cases} \frac{1}{2\sqrt{\theta}x} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Show that $X_{(n)}$ is sufficient for θ .
- (b) Consider a test of size α ($0 < \alpha < 1$) for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ ($> \theta_0$), that rejects H_0 if and only if $X_{(n)} > k$.
 - i. Determine the value of k .
 - ii. Find the minimum sample size required such that the power of the test is at least β ($\alpha < \beta < 1$).

8. Consider the regression model:

$$y_i = bx_i + e_i, \quad 1 \leq i \leq n,$$

where x_i 's are fixed non-zero real numbers and e_i 's are independent random variables with mean zero and equal variance.

- (a) Consider estimators of the form $\sum_{i=1}^n a_i y_i$ (where a_i 's are non random real numbers) that are unbiased for b . Show that the least squares estimator of b has the minimum variance in this class of estimators.
- (b) Suppose that x_i 's take values -1 or $+1$ and e_i 's have density

$$f(t) = \frac{1}{2} e^{-|t|}, \quad t \in R.$$

Find the maximum likelihood estimator of b .

9. Consider a collection of N cards, numbered $1, 2, \dots, N$, where $N \geq 2$. A card is drawn at random and set aside. Suppose that n cards are selected from the remaining $(N - 1)$ cards using SRSWR and their numbers noted as $Y_1, \dots, Y_n$. If $S = \sum_{i=1}^n Y_i$, find $E(S)$ and $\text{Var}(S)$.