2013

BOOKLET No. TEST CODE: PSB

*Afternoon*

**Questions: 10    Time: 2 hours**

*Write your Name, Registration number, Test Code, Number of this booklet, etc. in the appropriate places on the answer–booklet.*

- All questions carry equal weight.
- Answer at least one question from GROUP A.
- Best five answers subject to the above condition will be considered.

**Answer to each question should start on a fresh page.**

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR THE ANSWER–BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS.

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**STOP! WAIT FOR THE SIGNAL TO START.**

## GROUP A

1. Let $E = \{1, 2, \ldots, n\}$, where $n$ is an odd positive integer. Let $V$ be the vector space of all functions from $E$ to $\mathbb{R}^3$, where the vector space operations are given by

$$\begin{aligned}(f+g)(k) &= f(k)+g(k), \quad \text{for } f, g \in V, k \in E,\\ (\lambda f)(k) &= \lambda f(k), \quad \text{for } f \in V, \lambda \in \mathbb{R}, k \in E.\end{aligned}$$

   (a) Find the dimension of $V$.

   (b) Let $T\colon V \to V$ be the map given by

$$Tf(k) = \frac{1}{2}\left(f(k) + f(n+1-k)\right), \quad k \in E.$$

   Show that $T$ is linear.

   (c) Find the dimension of the null space of $T$.

2. Let $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$ be real numbers such that

$$\sum_{i=1}^{m} |a_i - x| = \sum_{j=1}^{n} |b_j - x| \text{ for all } x \in \mathbb{R}.$$

   Show that $m = n$ and $a_j = b_j$ for $1 \le j \le n$.

## GROUP B

3. Let $S = \{1, 2, \ldots, n\}$.

   (a) In how many ways can we choose two subsets $A$ and $B$ of $S$ so that $B \neq \emptyset$ and $B \subseteq A \subseteq S$?

   (b) In how many of these cases is $B$ a proper subset of $A$?

4. Consider a machine with three components whose times to failure are independently distributed as exponential random variables with mean $\lambda$. The machine continues to work as long as at least two components work. Find the expected time to failure of the machine.

5. A pin whose centre is fixed on a flat table is randomly and independently spun twice. Each time, the final position is noted by drawing a line segment.

   (a) What is the probability that the smallest angle between the two segments is more than half of the largest angle?

   (b) What is the probability that at least one of the two segments makes an angle which is less than $45°$ with the $x$-axis (when measured in the anti-clockwise direction)?

6. There are twenty individuals numbered $1, 2, \ldots, 20$. Each individual chooses 10 others from this group in a random fashion, independently of the choices of the others, and makes one phone call to each of the 10.

   (a) Let $X$ be the number of calls handled (incoming as well as outgoing) by Individual 1. Find $E(X)$.

   (b) Let $Y$ be the number of calls between Individual 1 and Individual 2. Find $E(Y)$.

   (c) Find $E(X|Y = 1)$.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $Uniform(\theta, 1)$ population, where $\theta < 1$.

   (a) Find the MLE $\hat{\theta}$ of $\theta$.

   (b) Find $c$ such that $c\hat{\theta}$ is unbiased for $\theta$.

8. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from some distribution with mean $\mu$ and variance $\sigma^2$. Let

$$s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

   where $\bar{X}$ is the sample mean. Show that $s$ underestimates $\sigma$ (that is, it has negative bias).

9. Let $X_1 \sim Geo(p_1)$ and $X_2 \sim Geo(p_2)$ be independent random variables, where $Geo(p)$ refers to the Geometric distribution whose p.m.f. $f$ is given by

$$f(k) = p(1-p)^k, \quad k = 0, 1, \ldots.$$

   We are interested in testing the null hypothesis $H_0 : p_1 = p_2$ against the alternative $H_1 : p_1 < p_2$. Intuitively, it is clear that we should reject if $X_1$ is large, but unfortunately we cannot compute a cutoff because the distribution of $X_1$ under $H_0$ depends on the unknown (common) value of $p_1$ and $p_2$.

   (a) Let $Y = X_1 + X_2$. Find the conditional distribution of $X_1|Y = y$ when $p_1 = p_2$.

   (b) Based on the result obtained in (a), derive a level 0.05 test for $H_0$ against $H_1$ that rejects $H_0$ when $X_1$ is large.

10. Let $y_1, y_2, y_3, y_4$ be uncorrelated observations with common variance $\sigma^2$ and expectations given by

$$E(y_1) = E(y_2) = \beta_1 + \beta_2 + \beta_3$$
$$E(y_3) = E(y_4) = \beta_1 - \beta_2$$

An observational function $\sum_{i=1}^{4} a_i y_i$ is said to be an *error function* if its expectation is zero.

(a) Obtain a maximal set of linearly independent error functions for the above model (that is, the set should be such that adding any other error function to the set would make the set linearly dependent). Justify your answer.

(b) Obtain an unbiased estimator of $3\beta_1 - \beta_2 + \beta_3$ such that it is uncorrelated with each of the error functions obtained in (a).