2012

BOOKLET No. TEST CODE: MS

Afternoon

Questions: 10	**Time: 2 hours**

Write your Name, Registration number, Test Code, Number of this booklet, etc. in the appropriate places on the answer–booklet.

- All questions carry equal weight.
- Answer at least one question from GROUP A.
- Best five answers subject to the above conditions will be considered.

Answer to each question should start on a fresh page.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR THE ANSWER–BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS.

STOP! WAIT FOR THE SIGNAL TO START.

GROUP A

1. Suppose V is the space of all $n \times n$ matrices with real elements. Define $T : V \to V$ by $T(A) = AB - BA$, $A \in V$, where $B \in V$ is a fixed matrix. Show that for any $B \in V$

   (a) T is linear;

   (b) T is not one-one;

   (c) T is not onto.

2. Let f be a real valued function satisfying

$$|f(x) - f(a)| \leq C |x - a|^{\gamma},$$

   for some $\gamma > 0$ and $C > 0$.

   (a) If $\gamma = 1$, show that f is continuous at a;

   (b) If $\gamma > 1$, show that f is differentiable at a.

GROUP B

3. Suppose integers are formed by taking one or more digits from the following

$$2, 2, 3, 3, 4, 5, 5, 5, 6, 7.$$

   For example, 355 is a possible choice while 44 is not. Find the number of distinct integers that can be formed in which

   (a) the digits are non-decreasing;

   (b) the digits are strictly increasing.

4. Consider n independent observations $\{(x_i, y_i) : 1 \le i \le n\}$ from the model

$$Y = \alpha + \beta x + \epsilon,$$

where ϵ is normal with mean 0 and variance σ^2. Let $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\sigma}^2$ be the maximum likelihood estimators of α, β and σ^2, respectively. Let v_{11}, v_{22} and v_{12} be the estimated values of $\mathrm{Var}(\hat{\alpha})$, $\mathrm{Var}(\hat{\beta})$ and $\mathrm{Cov}(\hat{\alpha}, \hat{\beta})$, respectively.

   (a) What is the estimated mean of Y when $x = x_0$? Estimate the mean squared error of this estimator.

   (b) What is the predicted value of Y when $x = x_0$? Estimate the mean squared error of this predictor.

5. A box has an unknown number of tickets serially numbered $1, 2, \ldots, N$. Two tickets are drawn using simple random sampling without replacement (SRSWOR) from the box. If X and Y are the numbers on these two tickets and $Z = \max(X, Y)$, show that

   (a) Z is not unbiased for N;

   (b) $aX + bY + c$ is unbiased for N if and only is $a + b = 2$ and $c = -1$.

6. Suppose X_1, X_2 and X_3 are three independent and identically distributed Bernoulli random variables with parameter p, $0 < p < 1$. Verify if the following statistics are sufficient for p:

   (a) $X_1 + 2X_2 + X_3$;

   (b) $2X_1 + 3X_2 + 4X_3$.

7. Suppose X_1 and X_2 are two independent and identically distributed random variables with Normal $(\theta, 1)$ distribution. Further, consider a Bernoulli random variable V with $P[V = 1] = 1/4$, which is independent of X_1 and X_2. Define X_3 as

$$X_3 = \begin{cases} X_1 & \text{when } V = 0, \\ X_2 & \text{when } V = 1. \end{cases}$$

For testing $H_0 : \theta = 0$ against $H_1 : \theta = 1$ consider the test:

$$\text{Reject } H_0 \text{ if } \left(X_1 + X_2 + X_3\right)/3 > c.$$

Find c such that the test has size 0.05.

8. Suppose X_1 is a standard normal random variable. Define

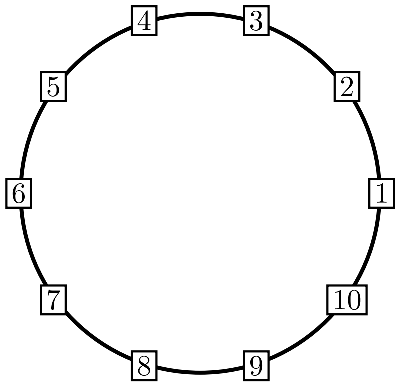
$$X_2 = \begin{cases} -X_1 & \text{if } |X_1| < 1, \\ X_1 & \text{otherwise.} \end{cases}$$

   (a) Show that X_2 is also a standard normal random variable.

   (b) Obtain the cumulative distribution function of $X_1 + X_2$ in terms of the cumulative distribution function of a standard normal random variable.

9. Envelopes are on sale for Rs. 30 each. Each envelope contains exactly one coupon, which can be one of four types with equal probability. Suppose you keep on buying envelopes and stop when you collect all the four types of coupons. What will be your expected expenditure?

10. There are 10 empty boxes numbered $1, 2, \ldots, 10$ placed sequentially on a circle as shown in the figure.



We perform 100 independent trials. At each trial, one box is selected with probability $1/10$ and one ball is placed in each of the two neighbouring boxes of the selected one.

Define X_k to be the number of balls in the k^{th} box at the end of 100 trials.

(a) Find $\mathrm{E}[X_k]$ for $1 \leq k \leq 10$.

(b) Find $\mathrm{Cov}(X_k, X_5)$ for $1 \leq k \leq 10$.