2016

BOOKLET NO. TEST CODE : **UGA**

Forenoon

Questions : 30 Time : 2 hours

Write your Name, Registration Number, Test Centre, Test Code and the Number of this Booklet in the appropriate places on the Answersheet.

This test contains 30 questions in all. For each of the 30 questions, there are four suggested answers. Only one of the suggested answers is correct. You will have to identify the correct answer in order to get full credit for that question. Indicate your choice of the correct answer by darkening the appropriate oval ⬬, completely on the answersheet.

You will get
4 marks for each correctly answered question,
0 marks for each incorrectly answered question and
1 mark for each unattempted question.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET ONLY.
YOU ARE NOT ALLOWED TO USE CALCULATOR.

WAIT FOR THE SIGNAL TO START.

1. The largest integer n for which $n + 5$ divides $n^5 + 5$ is

(A) 3115. (B) 3120. (C) 3125. (D) 3130.

2. Let p, q be primes and a, b be integers. If pa is divided by q, then the remainder is 1. If qb is divided by p, then also the remainder is 1. The remainder when $pa + qb$ is divided by pq is

(A) 1. (B) 0. (C) -1. (D) 2.

3. The polynomial $x^7 + x^2 + 1$ is divisible by

(A) $x^5 - x^4 + x^2 - x + 1$. (B) $x^5 + x^4 + 1$.

(C) $x^5 + x^4 + x^2 + x + 1$. (D) $x^5 - x^4 + x^2 + x + 1$.

4. Let $\alpha > 0$. If the equation $p(x) = x^3 - 9x^2 + 26x - \alpha$ has three positive real roots, then α must satisfy

(A) $\alpha \leq 27$. (B) $\alpha > 81$.

(C) $27 < \alpha \leq 54$. (D) $54 < \alpha \leq 81$.

5. The largest integer which is less than or equal to $(2 + \sqrt{3})^4$ is

(A) 192. (B) 193. (C) 194. (D) 195.

6. Consider a circle of unit radius and a chord of that circle that has unit length. The area of the largest triangle that can be inscribed in the circle with that chord as its base is

(A) $\frac{1}{2} + \frac{\sqrt{2}}{4}$. (B) $\frac{1}{2} + \frac{\sqrt{2}}{2}$.

(C) $\frac{1}{2} + \frac{\sqrt{3}}{4}$. (D) $\frac{1}{2} + \frac{\sqrt{3}}{2}$.

7. Let $z_1 = 3 + 4i$. If z_2 is a complex number such that $|z_2| = 2$, then the greatest and the least values of $|z_1 - z_2|$ are respectively

(A) 7 and 3. (B) 5 and 1.

(C) 9 and 5. (D) $4 + \sqrt{7}$ and $\sqrt{7}$.

8. Consider two distinct arithmetic progressions (AP) each of which has a positive first term and a positive common difference. Let S_n and T_n be the sums of the first n terms of these AP. Then $\lim_{n\to\infty} \frac{S_n}{T_n}$ equals

(A) ∞ or 0 depending on which AP has larger first term.

(B) ∞ or 0 depending on which AP has larger common difference.

(C) the ratio of the first terms of the AP.

(D) the ratio of the common differences of the AP.

9. Let $f(x) = \max\{\cos x, x^2\}$, $0 < x < \frac{\pi}{2}$. If x_0 is the solution of the equation $\cos x = x^2$ in $(0, \frac{\pi}{2})$, then

(A) f is continuous only at x_0.

(B) f is not continuous at x_0.

(C) f is continuous everywhere and differentiable only at x_0.

(D) f is differentiable everywhere except at x_0.

10. The set of all real numbers in $(-2, 2)$ satisfying

$$2^{|x|} - |2^{x-1} - 1| = 2^{x-1} + 1$$

is

(A) $\{-1, 1\}$. (B) $\{-1\} \cup [1, 2)$.

(C) $(-2, -1] \cup [1, 2)$. (D) $(-2, -1] \cup \{1\}$.

11. Let $S(k)$ denote the set of all one-to-one and onto functions from $\{1, 2, 3, \ldots, k\}$ to itself. Let p, q be positive integers. Let $S(p, q)$ be the set of all τ in $S(p+q)$ such that $\tau(1) < \tau(2) < \cdots < \tau(p)$ and $\tau(p+1) < \tau(p+2) < \cdots < \tau(p+q)$. The number of elements in the set $S(13, 29)$ is

(A) 377. (B) $(42)!$. (C) $\binom{42}{13}$. (D) $\frac{42!}{29!}$.

12. Suppose that both the roots of the equation $x^2 + ax + 2016 = 0$ are positive even integers. The number of possible values of a is

(A) 6. (B) 12. (C) 18. (D) 24.

13. Let $b \neq 0$ be a fixed real number. Consider the family of parabolas given by the equations

$$y^2 = 4ax + b, \quad \text{where } a \in \mathbb{R}.$$

The locus of the points on the parabolas at which the tangents to the parabolas make $45°$ angle with the x-axis is

(A) a straight line. (B) a pair of straight lines.
(C) a parabola. (D) a hyperbola.

14. Consider the curve represented by the equation

$$ax^2 + 2bxy + cy^2 + d = 0$$

in the plane, where $a > 0$, $c > 0$ and $ac > b^2$. Suppose that the normals to the curve drawn at 5 distinct points on the curve all pass through the origin. Then

(A) $a = c$ and $b > 0$. (B) $a \neq c$ and $b = 0$.
(C) $a \neq c$ and $b < 0$. (D) None of the above.

15. Let P be a 12-sided regular polygon and T be an equilateral triangle with its incircle having radius 1. If the area of P is the same as the area of T, then the length of the side of P is

(A) $\sqrt{\sqrt{3}\cot 15°}$. (B) $\sqrt{\sqrt{3}\tan 15°}$.

(C) $\sqrt{3\sqrt{2}\tan 15°}$. (D) $\sqrt{3\sqrt{2}\cot 15°}$.

16. Let ABC be a right-angled triangle with $\angle ABC = 90°$. Let P be the midpoint of BC and Q be a point on AB. Suppose that the length of BC is $2x$, $\angle ACQ = \alpha$, and $\angle APQ = \beta$. Then the length of AQ is

(A) $\dfrac{3x}{2\cot\alpha - \cot\beta}$. (B) $\dfrac{2x}{3\cot\alpha - 2\cot\beta}$.

(C) $\dfrac{3x}{\cot\alpha - 2\cot\beta}$. (D) $\dfrac{2x}{2\cot\alpha - 3\cot\beta}$.

17. Let $[x]$ denote the greatest integer less than or equal to x. The value of the integral

$$\int_1^n [x]^{x-[x]}\, dx$$

is equal to

(A) $1+\dfrac{2^3}{\log_e 2}-\dfrac{2^2}{\log_e 2}+\dfrac{3^4}{\log_e 3}-\dfrac{3^3}{\log_e 3}+\cdots+\dfrac{(n-1)^n}{\log_e(n-1)}-\dfrac{(n-1)^{n-1}}{\log_e(n-1)}$.

(B) $1+\dfrac{1}{\log_e 2}+\dfrac{2}{\log_e 3}+\cdots+\dfrac{n-2}{\log_e(n-1)}$.

(C) $\dfrac{1}{2}+\dfrac{2^2}{3}+\cdots+\dfrac{n^{n+1}}{n+1}$.

(D) $\dfrac{2^3-1}{3}+\dfrac{3^4-2^3}{4}+\cdots+\dfrac{n^{n+1}-(n-1)^n}{n+1}$.

18. Let $\alpha > 0, \beta \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous at 0 with $f(0) = \beta$. If $g(x) = |x|^{\alpha} f(x)$ is differentiable at 0, then

(A) $\alpha = 1$ and $\beta = 1$.

(B) $0 < \alpha < 1$ and $\beta = 0$.

(C) $\alpha \geq 1$ and $\beta = 0$.

(D) $\alpha > 0$ and $\beta > 0$.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable and strictly decreasing function such that $f(0) = 1$ and $f(1) = 0$. For $x \in \mathbb{R}$, let

$$F(x) = \int_0^x (t-2)f(t)\, dt.$$

Then

(A) F is strictly increasing in $[0, 3]$.

(B) F has a unique minimum in $(0, 3)$ but has no maximum in $(0, 3)$.

(C) F has a unique maximum in $(0, 3)$ but has no minimum in $(0, 3)$.

(D) F has a unique maximum and a unique minimum in $(0, 3)$.

20. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero function such that $\lim\limits_{x\to\infty} \dfrac{f(xy)}{x^3}$ exists for all $y > 0$. Let $g(y) = \lim\limits_{x\to\infty} \dfrac{f(xy)}{x^3}$. If $g(1) = 1$, then for all $y > 0$

(A) $g(y) = 1$.

(B) $g(y) = y$.

(C) $g(y) = y^2$.

(D) $g(y) = y^3$.

21. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Then the maximum number of points in D such that the distance between any pair of points is at least 1 will be

(A) 5. (B) 6. (C) 7. (D) 8.

22. The number of 3-digit numbers abc such that we can construct an isosceles triangle with sides a, b and c is

(A) 153. (B) 163. (C) 165. (D) 183.

23. The function

$$f(x) = x^{1/2} - 3x^{1/3} + 2x^{1/4}, \quad x \geq 0$$

(A) has more than two zeros.

(B) is always nonnegative.

(C) is negative for $0 < x < 1$.

(D) is one-to-one and onto.

24. Let $X = \{a + \sqrt{-5}\, b : a, b \in \mathbb{Z}\}$. An element $x \in X$ is called special if there exists $y \in X$ such that $xy = 1$. The number of special elements in X is

(A) 2. (B) 4. (C) 6. (D) 8.

25. For a set X, let $P(X)$ denote the set of all subsets of X. Consider the following statements.

(I) $P(A) \cap P(B) = P(A \cap B)$.
(II) $P(A) \cup P(B) = P(A \cup B)$.
(III) $P(A) = P(B) \implies A = B$.
(IV) $P(\emptyset) = \emptyset$.

Then

(A) All the statements are true.

(B) (I), (II), (III) are true and (IV) is false.

(C) (I), (III) are true and (II), (IV) are false.

(D) (II), (III), (IV) are true and (I) is false.

26. Let a, b, c be real numbers such that $a + b + c < 0$. Suppose that the equation $ax^2 + bx + c = 0$ has imaginary roots. Then

(A) $a < 0$ and $c < 0$. (B) $a < 0$ and $c > 0$.

(C) $a > 0$ and $c < 0$. (D) $a > 0$ and $c > 0$.

27. For $\alpha \in (0, \frac{3}{2})$, define $x_n = (n+1)^\alpha - n^\alpha$. Then $\lim_{n\to\infty} x_n$ is

(A) 1 for all α.

(B) 1 or 0 depending on the value of α.

(C) 1 or ∞ depending on the value of α.

(D) 1, 0, or ∞ depending on the value of α.

28. Let f be a continuous strictly increasing function from $[0, \infty)$ onto $[0, \infty)$ and $g = f^{-1}$ (that is, $f(x) = y$ if and only if $g(y) = x$). Let $a, b > 0$ and $a \neq b$. Then

$$\int_0^a f(x)\,dx + \int_0^b g(y)\,dy$$

is

(A) greater than or equal to ab.

(B) less than ab.

(C) always equal to ab.

(D) always equal to $\dfrac{af(a) + bg(b)}{2}$.

29. The sum of the series $\sum_{n=1}^{\infty} n^2 e^{-n}$ is

(A) $\dfrac{e^2}{(e-1)^3}$. (B) $\dfrac{e^2+e}{(e-1)^3}$. (C) $\dfrac{3}{2}$. (D) ∞.

30. Let $f : [0, 1] \to [-1, 1]$ be a non-zero function such that

$$f(2x) = 3f(x), \quad x \in [0, \tfrac{1}{2}].$$

Then $\lim_{x\to 0+} f(x)$ is equal to

(A) $\frac{1}{2}$. (B) $\frac{1}{3}$. (C) $\frac{2}{3}$. (D) 0.