NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 25, 2014

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions that follow.

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 12 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Calculus & Differential Equations and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded for questions involving more than one answer only if **all** the correct answers are given. **There will be no partial credit**.
- Calculators are **not allowed**.

Notation

- $\mathbb{N}$ denotes the set of natural numbers $\{1,2,3,\cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers.

- $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) denotes the n-dimensional Euclidean space over $\mathbb{R}$ (respectively, over $\mathbb{C}$), and is assumed to be endowed with its 'usual' topology. $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will denote the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and is identified with $\mathbb{R}^{n^2}$ (respectively, $\mathbb{C}^{n^2}$) when considered as a topological space.

- The symbol $]a,b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a,b]$ will stand for the corresponding closed interval; $[a,b[$ and $]a,b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.

- The space of continuous real-valued functions on an interval $[a,b]$ is denoted by $\mathcal{C}[a,b]$ and is endowed with its usual 'sup-norm' metric. The space of continuously differentiable real-valued functions on $[a,b]$ is denoted by $\mathcal{C}^1[a,b]$.

- The derivative of a function f is denoted by f' and the second derivative by f''.

- The transpose (respectively, adjoint) of a vector $x \in \mathbb{R}^n$ (respectively, $\mathbb{C}^n$) will be denoted by x^T (respectively, x^*). The transpose (respectively, adjoint) of a matrix $A \in \mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will be denoted by A^T (respectively, A^*).

- The symbol I will denote the identity matrix of appropriate order.

- The determinant of a square matrix A will be denoted by $\det(A)$ and its trace by $\mathrm{tr}(A)$.

- The null space of a linear functional φ (respectively, a linear operator A) on a vector space will be denoted by $\ker(\varphi)$ (respectively, $\ker(A)$). The range of the linear map A will be denoted by $\mathcal{R}(A)$.

- $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) will denote the group of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) with the group operation being matrix multiplication.

- The symbol S_n will denote the group of all permutations of n symbols $\{1,2,\cdots,n\}$, the group operation being composition.

- Unless specified otherwise, all logarithms are to the base e.

Section 1: Algebra

1.1 Let G be a finite group of order $n \geq 2$. Which of the following statements are true?
a. There always exists an injective homomorphism from G into S_n.
b. There always exists an injective homomorphism from G into S_m for some $m < n$.
c. There always exists an injective homomorphism from G into $GL_n(\mathbb{R})$.

1.2 Let $\mathbb{C}^*$ denote the multiplicative group of non-zero complex numbers and let P denote the subgroup of positive (real) numbers. Identify the quotient group $\mathbb{C}^*/P$.

1.3 Given a finite group and a prime p which divides its order, let $N(p)$ denote the number of p-Sylow subgroups of G. If G is a group of order 21, what are the possible values for $N(3)$ and $N(7)$?

1.4 Let V be the real vector space of all polynomials, in a single variable and with real coefficients, of degree at most 3. Let V^* be its dual space. Let $t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4$. Which of the following sets of functionals $\{f_i, 1 \leq i \leq 4\}$ form a basis for V^*?
a. For $1 \leq i \leq 4$, and for all $p \in V$, $f_i(p) = p(t_i)$.
b. For all $p \in V$, $f_i(p) = p(t_i)$ for $i = 1, 2$, $f_3(p) = p'(t_1)$ and $f_4(p) = p'(t_2)$.
c. For all $p \in V$, $f_i(p) = p(t_i)$ for $1 \leq i \leq 3$ and $f_4(p) = \int_1^2 p'(t)\, dt$.

1.5 Let V be a finite dimensional real vector space and let f and g be non-zero linear functionals on V. Assume that $\ker(f) \subset \ker(g)$. Which of the following statements are true?
a. $\ker(f) = \ker(g)$.
b. $f = \lambda g$ for some real number $\lambda \neq 0$.
c. The linear map $A : V \to \mathbb{R}^2$ defined by

$$Ax = (f(x), g(x)),$$

for all $x \in V$, is onto.

1.6 Let V be a finite dimensional real vector space and let $A : V \to V$ be a linear map such that $A^2 = A$. Assume that $A \neq 0$ and that $A \neq I$. Which of the following statements are true?
a. $\ker(A) \neq \{0\}$.
b. $V = \ker(A) \oplus \mathcal{R}(A)$.
c. The map $I + A$ is invertible.

1.7 Let $A \in \mathbb{M}_2(\mathbb{R})$ be a matrix which is not a diagonal matrix. Which of the following statements are true?
a. If $\mathrm{tr}(A) = -1$ and $\det(A) = 1$, then $A^3 = I$.
b. If $A^3 = I$, then $\mathrm{tr}(A) = -1$ and $\det(A) = 1$.
c. If $A^3 = I$, then A is diagonalizable over $\mathbb{R}$.

1.8 Let $x \in \mathbb{R}^n$ be a non-zero (column) vector. Define $A = xx^T \in \mathbb{M}_n(\mathbb{R})$.
a. What is the rank of A?
b. What is the necessary and sufficient condition for $I - 2A$ to be an orthogonal matrix?

1.9 Let $A \in GL_n(\mathbb{R})$ have integer entries. Let $b \in \mathbb{R}^n$ be a (column) vector, also with integer entries. Which of the following statements are true?
a. If $Ax = b$, then the entries of x are also integers.
b. If $Ax = b$, then the entries of x are rational.
c. The matrix A^{-1} has integer entries if, and only if, $\det(A) = \pm 1$.

1.10 In each of the following cases, describe the smallest subset of $\mathbb{C}$ which contains all the eigenvalues of every member of the set S.
a. $S = \{A \in \mathbb{M}_n(\mathbb{C}) \mid A = BB^* \text{ for some } B \in \mathbb{M}_n(\mathbb{C})\}$.
b. $S = \{A \in \mathbb{M}_n(\mathbb{C}) \mid A = B + B^* \text{ for some } B \in \mathbb{M}_n(\mathbb{C})\}$.
c. $S = \{A \in \mathbb{M}_n(\mathbb{C}) \mid A + A^* = 0\}$.

Section 2: Analysis

2.1 Find the largest interval for which the following series is convergent at all points x in it.

$$\sum_{n=1}^{\infty} \frac{2^n(3x-1)^n}{n}.$$

2.2 Let m and k be fixed positive integers. Evaluate:

$$\lim_{n\to\infty} \left(\frac{(n+1)^m + (n+2)^m + \cdots + (n+k)^m}{n^{m-1}} - kn \right).$$

2.3 Which of the following statements are true?
a. If f is twice continuously differentiable in $]a, b[$ and if for all $x \in]a, b[$,

$$f''(x) + 2f'(x) + 3f(x) \;=\; 0,$$

then f is infinitely differentiable in $]a, b[$.
b. Let $f \in \mathcal{C}[a, b]$ be differentiable in $]a, b[$. If $f(a) = f(b) = 0$, then, for any real number α, there exists $x \in]a, b[$ such that

$$f'(x) + \alpha f(x) \;=\; 0.$$

c. The function defined below is not differentiable at $x = 0$.

$$f(x) \;=\; \begin{cases} x^2 |\cos \frac{\pi}{x}|, & x \neq 0, \\ 0 & x = 0. \end{cases}$$

2.4 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Which of the following statements are true?
a. If f is bounded, then f is uniformly continuous.
b. If f is differentiable and if f' is bounded, then f is uniformly continuous.
c. If $\lim_{|x|\to\infty} f(x) = 0$, then f is uniformly continuous.

2.5 In which of the following cases, is the function f of bounded variation on $[0, 1]$?
a. The function $f : [0, 1] \to \mathbb{R}$ such that, for all $x, y \in [0, 1]$,

$$|f(x) - f(y)| \;\leq\; 3|x - y|.$$

b. The function f is monotonically decreasing on $[0, 1]$.
c. If for some non-negative Riemann integrable function g on $[0, 1]$,

$$f(x) \;=\; \int_0^x g(t)\, dt \text{ for all } x \in [0, 1].$$

2.6 Let $g_n(x) \;=\; n[f(x + \frac{1}{n}) - f(x)]$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Which of the following statements are true?
a. If $f(x) = x^3$, then $g_n \to f'$ uniformly on $\mathbb{R}$ as $n \to \infty$.
b. If $f(x) = x^2$, then $g_n \to f'$ uniformly on $\mathbb{R}$ as $n \to \infty$.
c. If f is differentiable and if f' is uniformly continuous on $\mathbb{R}$, then $g_n \to f'$ uniformly on $\mathbb{R}$ as $n \to \infty$.

2.7 Which of the following statements are true?
a. The series

$$\sum_{n=1}^{\infty} \frac{x^2}{1+n^2x^2}$$

does not converge uniformly on $\mathbb{R}$.
b. The series in (a) above converges uniformly on $\mathbb{R}$.
c. The sum of the series

$$\sum_{n=1}^{\infty} \frac{\sin nx^2}{1+n^3}$$

defines a continuously differentiable function on $\mathbb{R}$.

2.8 Find the sum of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}.$$

2.9 Let $\{f_n\}$ be a sequence of bounded real valued functions on $[0,1]$ converging to f at all points of this interval. Which of the following statements are true?
a. If f_n and f are all continuous, then

$$\lim_{n\to\infty} \int_0^1 f_n(t)\ dt \ = \ \int_0^1 f(t)\ dt.$$

b. If $f_n \to f$ uniformly, as $n \to \infty$, on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(t)\ dt \ = \ \int_0^1 f(t)\ dt.$$

c. If $\int_0^1 |f_n(t) - f(t)|\ dt \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int_0^1 f_n(t)\ dt \ = \ \int_0^1 f(t)\ dt.$$

2.10 Let $f : [0,\pi] \to \mathbb{R}$ be a continuous function such that $f(0) = 0$. Which of the following statements are true?
a. If

$$\int_0^\pi f(t)\cos nt\ dt \ = \ 0$$

for all $n \in \{0\} \cup \mathbb{N}$, then $f \equiv 0$.
b. If

$$\int_0^\pi f(t)\sin nt\ dt \ = \ 0$$

for all $n \in \mathbb{N}$, then $f \equiv 0$.
c. If

$$\int_0^\pi t^n f(t)\ dt \ = \ 0$$

for all $n \in \{0\} \cup \mathbb{N}$, then $f \equiv 0$.

Section 3: Topology

3.1 Let A and B be subsets of $\mathbb{R}^n$. Define

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Consider the sets

$$\begin{aligned} W &= \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}, \\ X &= \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, y = 0\}, \\ Y &= \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}, \\ Z &= \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1, |y| \leq 1\}. \end{aligned}$$

Which of the following statements are true?
a. The set $W + X$ is open.
b. The set $X + Y$ is closed.
c. The set $Y + Z$ is closed.

3.2 Let X be a topological space and let A be a subset of X. Which of the following statements are true?
a. If A is dense in X, then A° (the interior of A), is also dense in X.
b. If A is dense in X, then $X \backslash A$ is nowhere dense.
c. If A is nowhere dense, then $X \backslash A$ is dense.

3.3 Consider the space $X = \mathcal{C}[0, 1]$ with its usual 'sup-norm' topology. Let

$$S = \left\{ f \in X \mid \int_0^1 f(t)\, dt \neq 0 \right\}.$$

Which of the following statements are true?
a. The set S is open.
b. The set S is dense in X.
c. The set S is connected.

3.4 Consider the space $X = \mathcal{C}[0, 1]$ with its usual 'sup-norm' topology. Let

$$S = \left\{ f \in X \mid \int_0^1 f(t)\, dt = 0 \right\}.$$

Which of the following statements are true?
a. The set S is closed.
b. The set S is connected.
c. The set S is compact.

3.5 Let (X, d) be a metric space. Which of the following statements are true?
a. A sequence $\{x_n\}$ converges to x in X if, and only if, the sequence $\{y_n\}$ is a Cauchy sequence in X, where, for $k \geq 1$, $y_{2k-1} = x_k$ and $y_{2k} = x$.
b. If $f : X \to X$ maps Cauchy sequences into Cauchy sequences, then f is continuous.
c. If $f : X \to X$ is continuous, then it maps Cauchy sequences into Cauchy sequences.

3.6 Which of the following spaces are separable?
a. The space $\mathcal{C}[a,b]$ with its usual 'sup-norm' topology.
b. The space $\mathcal{C}[0,1]$ with the metric defined by

$$d(f,g) \;=\; \int_0^1 |f(t)-g(t)|\, dt.$$

c. The space ℓ_∞ consisting of all bounded real sequences with the metric

$$d(\{x_n\},\{y_n\}) \;=\; \sup_{n\in\mathbb{N}} |x_n - y_n|.$$

3.7 Consider the space $\mathbb{M}_2(\mathbb{R})$ with its usual topology. Which of the following sets are dense?
a. The set of all invertible matrices.
b. The set of all matrices with both eigenvalues real.
c. The set of all matrices A such that $\mathrm{tr}(A) = 0$.

3.8 Which of the following statements are true?
a. If $f : \mathbb{R} \to \mathbb{R}$ is injective and continuous, then it is strictly monotonic.
b. If $f \in \mathcal{C}[0,2]$ is such that $f(0) = f(2)$, then there exist x_1 and x_2 in $[0,2]$ such that $x_1 - x_2 = 1$ and $f(x_1) = f(x_2)$.
c. Let f and g be continuous real valued functions on $\mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $f(g(x)) = g(f(x))$. If there exists $x_0 \in \mathbb{R}$ such that $f(f(x_0)) = g(g(x_0))$, then there exists $x_1 \in \mathbb{R}$ such that $f(x_1) = g(x_1)$.

3.9 Which of the following statements are true?
a. Let $V = \mathcal{C}_c(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ with compact support (*i.e.* each function vanishes outside a compact set) endowed with the metric

$$d(f,g) \;=\; \left(\int_{-\infty}^{\infty} |f(t)-g(t)|^2\, dt\right)^{\frac{1}{2}}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which vanishes outside the interval $[0,1]$. Define $f_n(x) = f(x-n)$ for $n \in \mathbb{N}$. Then $\{f_n\}$ has a convergent subsequence in V.
b. Let φ, ψ be continuous functions on $[0,1]$. Let $\{f_n\}$ be a sequence in $\mathcal{C}[0,1]$ with its usual 'sup-norm' topology such that, for all $n \in \mathbb{N}$, the functions f_n are continuously differentiable and for all $x \in [0,1]$, and for all $n \in \mathbb{N}$ we have $|f_n(x)| \le \varphi(x)$ and $|f_n'(x)| \le \psi(x)$. Then there exists a subsequence of $\{f_n\}$ which converges in $\mathcal{C}[0,1]$.
c. Let $\{A_n\}$ be a sequence of orthogonal matrices in $\mathbb{M}_2(\mathbb{R})$. Then it has a convergent subsequence.

3.10 Which of the following pairs of sets are homeomorphic?
a. The sets $\mathbb{Q}$ and $\mathbb{Z}$ with their usual topologies inherited from $\mathbb{R}$.
b. The sets $]0,1[$ and $]0,\infty[$ with their usual topologies inherited from $\mathbb{R}$.
c. The sets $S^1 = \{z \in \mathbb{C} \mid z = e^{i\theta}, 0 \le \theta < 2\pi\}$ and $A = \{z \in \mathbb{C} \mid z = re^{i\theta}, 1 \le r \le 2,\ 0 \le \theta < 2\pi\}$ with their usual topologies inherited from $\mathbb{C} \cong \mathbb{R}^2$.

Section 4: Calculus and Differential Equations

4.1 Let

$$S = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}.$$

Find the volume of the set S.

4.2 Let $F :]0, \infty[\to \mathbb{R}$ be defined by:

$$F(x) = \int_{-x}^{x} \frac{1 - e^{-xy}}{y} \, dy.$$

Compute $F'(x)$.

4.3 Let $f(x, y) = x^2 + 5y^2 - 6x + 10y + 6$. Where are the maxima/minima of f (if any) located?

4.4 Evaluate:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(2x^2+2xy+2y^2)} \, dx dy.$$

4.5 Write down the Taylor series expansion about the origin, up to the term involving x^7, for the function

$$f(x) = \frac{1}{2}[x\sqrt{1 - x^2} + \sin^{-1} x].$$

4.6 Solve:

$$\begin{aligned} -\tfrac{d^2u}{dr^2} - \tfrac{1}{r}\tfrac{du}{dr} &= 1, \text{ in } 0 < r < 1, \\ u'(0) = 0 &= u(1). \end{aligned}$$

4.7 Which of the following two-point boundary value problems admit a unique solution?
a. $-u''(x) = 2x$ in $0 < x < 1$ and $u(0) = u(1) = 0$.
b. $-u''(x) = 2x$ in $0 < x < 1$ and $u(0) = u'(1) = 0$.
c. $-u''(x) = 2x$ in $0 < x < 1$ and $u'(0) = u'(1) = 0$.

4.8 Which of the following statements are true?
a. Let ψ be a non-negative and continuously differentiable function on $]0, \infty[$ such that $\psi'(x) \leq \psi(x)$ for all $x \in]0, \infty[$. Then

$$\lim_{x \to \infty} \psi(x) = 0.$$

b. Let ψ be a non-negative function continuous on $[0, \infty[$ and differentiable on $]0, \infty[$ such that $\psi(0) = 0$ and such that $\psi'(x) \leq \psi(x)$ for all $x \in]0, \infty[$. Then $\psi \equiv 0$.
c. Let φ be a non-negative and continuous function on $[0, \infty[$ and such that

$$\varphi(x) \leq \int_0^x \varphi(t) \, dt$$

for all $x \in [0, \infty[$. Then $\varphi \equiv 0$.

4.9 Write down the expression for the Laplace transform $F(s)$ of the function $f(x) = x^n$, where $n \in \mathbb{N}$.

4.10 Amongst all smooth curves $y(x)$ passing through the points $(x_1, 0)$ and $(x_2, 0)$ in the plane, we wish to find that whose surface of revolution about the x-axis has the least surface area. Write down the functional that must be minimised to find this curve.

Section 5: Miscellaneous

5.1 Let $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$ be defined by

$$a_{ij} = \begin{cases} i, & \text{if } i+j = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\det(A)$.

5.2 Let $n \in \mathbb{N}$ be fixed. For $0 \leq k \leq n$, let C_k denote the usual binomial coefficient $\left[\begin{array}{c} n \\ k \end{array}\right]$ of choosing k objects from a set of n objects. Evaluate:

$$C_0^2 + C_1^2 + \cdots + C_n^2.$$

5.3 Which of the following numbers are prime?
a. 179.
b. 197.
c. 199.

5.4 Given $f : \mathbb{R} \to \mathbb{R}$, define $f^2(x) = f(f(x))$. Which of the following statements are true?
a. If f is strictly monotonic, then f^2 is strictly increasing.
b. If $f^2(x) = -x$ for all $x \in \mathbb{R}$, then f is injective.
c. There does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f^2(x) = -x$ for all $x \in \mathbb{R}$.

5.5 Let a be a fixed positive real number. Evaluate:

$$\max_{\substack{x_i \geq 0, 1 \leq i \leq n \\ \sum_{i=1}^n x_i = a}} x_1 x_2 \cdots x_n.$$

5.6 A real number is said to be *algebraic* if it is the root of a non-zero polynomial of degree at least one with integer coefficients. Otherwise the number is said to be *transcendental*. Which of the following statements are true?
a. Algebraic numbers are dense in $\mathbb{R}$.
b. Transcendental numbers are dense in $\mathbb{R}$.
c. The number $\cos(\frac{\pi}{13})$ is algebraic.

5.7 Let $f :]a, b[\to \mathbb{R}$ be a given function. Which of the following statements are true?
a. If f is convex in $]a, b[$, then the set

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x \in]a, b[,\ y \geq f(x)\}$$

is a convex set.
b. If f is convex in $]a, b[$, then the set

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x \in]a, b[,\ y \leq f(x)\}$$

is a convex set.
c. If f is convex in $]a, b[$, then $|f|$ is also convex in $]a, b[$.

5.8 Two fair dice are rolled. What is the probability that the sum of the numbers on the top faces is 8?

5.9 Let $\{f_n\}_{n=1}^{\infty}$ and f be real valued functions defined on $\mathbb{R}$. For $\varepsilon > 0$ and for $m \in \mathbb{N}$, define

$$E_m(\varepsilon) \;=\; \{x \in \mathbb{R} \mid |f_m(x) - f(x)| \geq \varepsilon\}.$$

Let

$$S \;=\; \{x \in \mathbb{R} \mid \text{the sequence } \{f_n(x)\} \text{ does not converge to } f(x)\}.$$

Express S in terms of the sets $\{E_m(\varepsilon)\}_{m\in\mathbb{N},\varepsilon>0}$ (using the set theoretic operations of unions and intersections).

5.10 Consider the Fibonacci sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$a_0 \;=\; a_1 \;=\; 1 \text{ and } a_n \;=\; a_{n-1} + a_{n-2}, \;\; n \geq 2.$$

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function. Express F in closed form as a function of z.