NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 24, 2009

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 12 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.

- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.

- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.

- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).

- Points will be awarded in the above questions only if **all** the correct choices are made. **There will be no partial credit**.

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, which is assumed to be endowed with its 'usual' topology. The symbol $\mathbb{Z}_n$ will denote the ring of integers modulo n. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. The space of continuous real valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup' norm.

Section 1: Algebra

1.1 Pick out the cases where the given subgroup H is a normal subgroup of the group G.
(a) G is the group of all 2×2 invertible upper triangular matrices with real entries, under matrix multiplication, and H is the subgroup of all such matrices (a_{ij}) such that $a_{11} = 1$.
(b) G is the group of all 2×2 invertible upper triangular matrices with real entries, under matrix multiplication, and H is the subgroup of all such matrices (a_{ij}) such that $a_{11} = a_{22}$.
(c) G is the group of all $n \times n$ invertible matrices with real entries, under matrix multiplication, and H is the subgroup of such matrices with positive determinant.

1.2 Let $GL(n, \mathbb{R})$ denote the group of all invertible $n \times n$ matrices with real entries, under matrix multiplication, and let $SL(n, \mathbb{R})$ denote the subgroup of such matrices whose determinant is equal to unity. Identify the quotient group $GL(n, \mathbb{R})/SL(n, \mathbb{R})$.

1.3 Let S_n denote the symmetric group of permutations of n symbols. Does S_7 contain an element of order 10? If 'yes', write down an example of such an element.

1.4 What is the largest possible order of an element in S_7?

1.5 Write down all the units in the ring $\mathbb{Z}_8$ of all integers modulo 8.

1.6 Pick out the cases where the given ideal is a maximal ideal.
(a) The ideal $15\mathbb{Z}$ in $\mathbb{Z}$.
(b) The ideal $\mathcal{I} = \{f \ : \ f(0) = 0\}$ in the ring $\mathcal{C}[0, 1]$ of all continous real valued functions on the interval $[0, 1]$.
(c) The ideal generated by $x^3 + x + 1$ in the ring of polynomials $\mathbb{F}_3[x]$, where $\mathbb{F}_3$ is the field of three elements.

1.7 Let A be a 2×2 matrix with complex entries which is non-zero and non-diagonal. Pick out the cases when A is diagonalizable.
(a) $A^2 = I$.
(b) $A^2 = 0$.
(c) All eigenvalues of A are equal to 2.

1.8 Let $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$ be two non-zero (column) vectors. Let $\mathbf{y}^T$ denote the transpose of $\mathbf{y}$. Let $A = \mathbf{x}\mathbf{y}^T$, *i.e.* $A = (a_{ij})$ where $a_{ij} = x_i y_j$. What is the rank of A?

1.9 Let $\mathbf{x}$ be a non-zero (column) vector in $\mathbb{R}^n$. What is the necessary and sufficient condition for the matrix $A = I - 2\mathbf{x}\mathbf{x}^T$ to be orthogonal?

1.10 Let A be an $n \times n$ matrix with complex entries. Pick out the true statements.
(a) A is always similar to an upper-triangular matrix.
(b) A is always similar to a diagonal matrix.
(c) A is similar to a block diagonal matrix, with each diagonal block of size strictly less than n, provided A has at least 2 distinct eigenvalues.

Section 2: Analysis

2.1 Evaluate:

$$\lim_{n\to\infty} n\sin(2\pi e n!).$$

2.2 Evaluate:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{4n^2 - k^2}}.$$

2.3 Pick out the convergent series:

(a)

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+\frac{3}{2}}}.$$

(b)

$$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots$$

(c)

$$\sum_{n=1}^{\infty} \sqrt{\frac{1+4^n}{1+5^n}}.$$

2.4 Which of the following functions are continuous?

(a)

$$f(x) \;=\; x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots,\ x \in \mathbb{R}.$$

(b)

$$f(x) \;=\; \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^{\frac{3}{2}}},\ x \in [-\pi, \pi].$$

(c)

$$f(x) \;=\; \sum_{n=1}^{\infty} n^2 x^n,\ x \in [-\frac{1}{2}, \frac{1}{2}].$$

2.5 Which of the following functions are uniformly continuous?

(a) $f(x) = \frac{1}{x}$ in $(0, 1)$.
(b) $f(x) = x^2$ in $\mathbb{R}$.
(c) $f(x) = \sin^2 x$ in $\mathbb{R}$.

2.6 Pick out the sequences $\{f_n\}$ which are uniformly convergent.

(a)

$$f_n(x) \;=\; nxe^{-nx} \;\text{ on }\; (0, \infty).$$

(b)

$$f_n(x) \;=\; x^n \;\text{ on }\; [0, 1].$$

(c)

$$f_n(x) \;=\; \frac{\sin nx}{\sqrt{n}} \;\text{ on }\; \mathbb{R}.$$

2.7 Which of the following functions are Riemann integrable on the interval $[0, 1]$?

(a)

$$f(x) = \lim_{n\to\infty} \cos^{2n}(24\pi x).$$

(b)

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

(c)

$$f(x) = \begin{cases} \cos x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \sin x, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

2.8 Let $z = x + iy$ be a complex number, where x and $y \in \mathbb{R}$, and let $f(z) = u(x, y) + iv(x, y)$, where u and v are real valued, be an analytic function on $\mathbb{C}$. Express $f'(z)$ in terms of the partial derivatives of u and v.

2.9 Let $z \in \mathbb{C}$ be as in the previous question. Find the image of the set $S = \{z : x > 0, 0 < y < 2\}$ under the transformation $f(z) = iz + 1$.

2.10 Find the residue at $z = 0$ for the function

$$f(z) = \frac{1 + 2z}{z^2 + z^3}.$$

Section 3: Topology

3.1 Let X be a metric space and let $f : X \to \mathbb{R}$ be a continuous function. Pick out the true statements.
(a) f always maps Cauchy sequences into Cauchy sequences.
(b) If X is compact, then f always maps Cauchy sequences into Cauchy sequences.
(c) If $X = \mathbb{R}^n$, then f always maps Cauchy sequences into Cauchy sequences.

3.2 Let B be the closed ball in $\mathbb{R}^2$ with centre at the origin and radius unity. Pick out the true statements.
(a) There exists a continuous function $f : B \to \mathbb{R}$ which is one-one.
(b) There exists a continuous function $f : B \to \mathbb{R}$ which is onto.
(a) There exists a continuous function $f : B \to \mathbb{R}$ which is one-one and onto.

3.3 Let A and B be subsets of $\mathbb{R}$. Define

$$C = \{a + b : a \in A, b \in B\}.$$

Pick out the true statements.
(a) C is closed if A and B are closed.
(b) C is closed if A is closed and B is compact.
(c) C is compact if A is closed and B is compact.

3.4 Which of the following subsets of $\mathbb{R}^2$ are compact?
(a) $\{(x, y) : xy = 1\}$
(b) $\{(x, y) : x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1\}$
(c) $\{(x, y) : x^2 + y^2 < 1\}$

3.5 Which of the following sets in $\mathbb{R}^2$ are connected?
(a) $\{(x, y) : x^2y^2 = 1\}$
(b) $\{(x, y) : x^2 + y^2 = 1\}$
(c) $\{(x, y) : 1 < x^2 + y^2 < 2\}$

3.6 Let $\mathcal{P}$ denote the set of all polynomials in the real variable x which varies over the interval $[0, 1]$. What is the closure of $\mathcal{P}$ in $\mathcal{C}[0, 1]$ (with its usual sup-norm topology)?

3.7 Let $\{f_n\}$ be a sequence of functions which are continuous over $[0,1]$ and continuously differentiable in $]0,1[$. Assume that $|f_n(x)| \leq 1$ and that $|f_n'(x)| \leq 1$ for all $x \in]0,1[$ and for each positive integer n. Pick out the true statements.
(a) f_n is uniformly continuous for each n.
(b) $\{f_n\}$ is a convergent sequence in $\mathcal{C}[0,1]$.
(c) $\{f_n\}$ contains a subsequence which converges in $\mathcal{C}[0,1]$.

3.8 Pick out the true statements.
(a) Let $f : [0,2] \to [0,1]$ be a continuous function. Then, there always exists $x \in [0,1]$ such that $f(x) = x$.
(b) Let $f : [0,1] \to [0,1]$ be a continuous function which is continuously differentiable in $]0,1[$ and such that $|f'(x)| \leq \frac{1}{2}$ for all $x \in]0,1[$. Then, there exists a unique $x \in [0,1]$ such that $f(x) = x$.
(c) Let $S = \{\mathbf{p} = (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Let $f : S \to S$ be a continuous function. Then, there always exists $\mathbf{p} \in S$ such that $f(\mathbf{p}) = \mathbf{p}$.

3.9 Let (X,d) be a metric space. Let A and B be subsets of X. Define

$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}.$$

For $x \in X$, define

$$d(x,A) = \inf\{d(x,a) : a \in A\}.$$

Pick out the true statements.
(a) The function $x \mapsto d(x,A)$ is uniformly continuous.
(b) $d(x,A) = 0$ if, and only if, $x \in A$.
(c) $d(A,B) = 0$ implies that $A \cap B \neq \emptyset$.

3.10 Let

$$B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \text{ and } D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

Pick out the true statements.
(a) Given a continuous function $g : B \to \mathbb{R}$, there always exists a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f = g$ on B.
(b) Given a continuous function $g : D \to \mathbb{R}$, there always exists a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f = g$ on D.
(c) There exists a continous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \equiv 1$ on the set $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = \frac{3}{2}\}$ and $f \equiv 0$ on the set $B \cup \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \geq 2\}$.

Section 4: Applied Mathematics

4.1 A spherical ball of volatile material evaporates (*i.e.* its volume decreases) at a rate proportional to its surface area. If the initial radius is r_0 and at time $t = 1$, the radius is $r_0/2$, find the time at which the ball disappears completely.

4.2 A body of mass m falling from rest under gravity experiences air resistance proportional to the square of its velocity. Write down the initial value problem for the vertical displacement x of the body.

4.3 A body falling from rest under gravity travels a distance y and has a velocity v at time t. Write down the relationship between v and y.

4.4 Let $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and let $\mathbf{v}(\mathbf{x}) = \mathbf{x}$. Apply Gauss' divergence theorem to $\mathbf{v}$ over the unit ball

$$B = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \leq 1\}$$

and deduce the relationship between ω_n, the (n-dimensional) volume of B, and σ_n, the (($n-1$)-dimensional) surface measure of B.

4.5 Write down the general solution of the linear system:

$$\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 4x - 2y. \end{aligned}$$

4.6 What is the smallest positive value of λ such that the problem:

$$\begin{aligned} u'' + \lambda u &= 0 \text{ in }]0,1[\\ u(0) = u(1) \quad &\text{and} \quad u'(0) = u'(1) \end{aligned}$$

has a solution such that $u \not\equiv 0$?

4.7 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Write down the solution of the problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0,\ t > 0, \\ u(x,0) &= f(x),\ x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(x,0) &= 0,\ x \in \mathbb{R}. \end{aligned}$$

4.8 Use duality to find the optimal value of the cost function in the following linear programming problem:

$$\begin{gathered} Max.\ x + y + z \\ \text{such that } 3x + 2y + 2z = 1, \\ x \geq 0,\ y \geq 0,\ z \geq 0. \end{gathered}$$

4.9 The value of $\sqrt{10}$ is computed by solving the equation $x^2 = 10$ using the Newton-Raphson method. Starting from some value $x_0 > 0$, write down the iteration scheme.

4.10 Write down the Laplace transform $F(s)$ of the function $f(x) = x^3$.

Section 5: Miscellaneous

5.1 Let $P = (0, 1)$ and $Q = (4, 1)$ be points on the plane. Let A be a point which moves on the x-axis between the points $(0, 0)$ and $(4, 0)$. Let B be a point which moves on the line $y = 2$ between the points $(0, 2)$ and $(4, 2)$. Consider all possible paths consisting of the line segments PA, AB and BQ. What is the shortest possible length of such a path?

5.2 A convex polygon has its interior angles in arithmetic progression, the least angle being $120°$ and the common difference being $5°$. Find the number of sides of the polygon.

5.3 Let a, b and c be the lengths of the sides of an arbitrary triangle. Define

$$x = \frac{ab + bc + ca}{a^2 + b^2 + c^2}.$$

Pick out the true statements.
(a) $\frac{1}{2} \leq x \leq 2$.
(b) $\frac{1}{2} \leq x \leq 1$.
(c) $\frac{1}{2} < x \leq 1$.

5.4 What is the maximum number of pieces that can be obtained from a pizza by making 7 cuts with a knife?

5.5 In arithmetic base 3, a number is expressed as 210100. Find its square root and express it in base 3.

5.6 Evaluate:

$$\left(\frac{-1 + i\sqrt{3}}{\sqrt{2} + i\sqrt{2}} \right)^{20}.$$

5.7 Let n be a fixed positive integer and let C_r stand for the usual binomial coefficients *i.e.*, the number of ways of choosing r objects from n objects. Evaluate:

$$C_1 + 2C_2 + \cdots nC_n.$$

5.8 Let x, y and z be real numbers such that $x^2 + y^2 + z^2 = 1$. Find the maximum and minimum values of $2x + 3y + z$.

5.9 Let $x_0 = 0$. For $n \geq 0$, define

$$x_{n+1} \;=\; x_n^2 + \frac{1}{4}.$$

Pick out the true statements:
(a) The sequence $\{x_n\}$ is bounded.
(b) The sequence $\{x_n\}$ is monotonic.
(a) The sequence $\{x_n\}$ is convergent.

5.10 Seven tickets are numbered consecutively from 1 to 7. Two of them are selected in order without replacement. Let A denote the event that the numbers on the two tickets add up to 9. Let B be the event that the numbers on the two tickets differ by 3. If each draw has equal probability $\frac{1}{42}$ (the draw $(1, 7)$ being considered as distinct from the draw $(7, 1)$, for example) find the probabilty $P(B|A)$.