NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Awards Screening Test

February 25, 2006

Time Allowed: 90 Minutes

Maximum Marks: 40

Please read, carefully, the instructions on the following page before you write anything on this booklet

NAME:	ROLL No.:
Institution	

(For Official Use)

Sec. 1	Sec. 2	Sec. 3	Sec. 4	Sec. 5	TOTAL

INSTRUCTIONS TO CANDIDATES

- Do not forget to write your name and roll number on the cover page. In the box marked 'Institution', fill in the name of the institution where you are working towards a Ph.D. degree. In case you have not yet joined any institution for research, write *Not Applicable*.

- Please ensure that your answer booklet contains 16 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.

- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum marks to be scored is **forty**.

- Answer each question, as directed, in the space provided at the end of it. Answers are to be given in the form of a word (or words, if required), a numerical value (or values) or a simple mathematical expression. **Do not write sentences.**

- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), (c) and (d)).

- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, which is assumed to be endowed with its 'usual' topology. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order.

Section 1: Algebra

1.1 Let $f : (\mathbb{Q}, +) \to (\mathbb{Q}, +)$ be a non-zero homomorphism. Pick out the true statements:
a. f is always one-one.
b. f is always onto.
c. f is always a bijection.
d. f need be neither one-one nor onto.

Answer:

1.2 Consider the element

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$$

of the symmetric group S_5 on five elements. Pick out the true statements:
a. The order of α is 5.
b. α is conjugate to

$$\begin{pmatrix} 4 & 5 & 2 & 3 & 1 \\ 5 & 4 & 3 & 1 & 2 \end{pmatrix}.$$

c. α is the product of two cycles.
d. α commutes with all elements of S_5.

Answer:

1.3 Let G be a group of order 60. Pick out the true statements:
a. G is abelian.
b. G has a subgroup of order 30.
c. G has subgroups of order 2,3 and 5.
d. G has subgroups of order 6, 10 and 15.

Answer:

1.4 Consider the polynomial ring $R[x]$ where $R = \mathbb{Z}/12\mathbb{Z}$ and write the elements of R as $\{0, 1, \cdots, 11\}$. Write down all the distinct roots of the polynomial $f(x) = x^2 + 7x$ of $R[x]$.

Answer:

1.5 Let R be the polynomial ring $\mathbb{Z}_2[x]$ and write the elements of $\mathbb{Z}_2$ as $\{0, 1\}$. Let $(f(x))$ denote the ideal generated by the element $f(x) \in R$. If $f(x) = x^2 + x + 1$, then the quotient ring $R/(f(x))$ is
a. a ring but not an integral domain.
b. an integral domain but not a field.
c. a finite field of order 4.
d. an infinite field.

Answer:

1.6 Consider the set of all linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^4$ over $\mathbb{R}$. What is the dimension of this set, considered as a vector space over $\mathbb{R}$ with pointwise operations?

Answer:

1.7 Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$. Write down a matrix P such that $P^{-1}AP$ is a diagonal matrix.

Answer : $P =$

1.8 Let A be an orthogonal 3×3 matrix with real entries. Pick out the true statements:
a. The determinant of A is a rational number.
b. $d(Ax, Ay) = d(x, y)$ for any two vectors x and $y \in \mathbb{R}^3$, where $d(u, v)$ denotes the usual Euclidean distance between vectors u and $v \in \mathbb{R}^3$.
c. All the entries of A are positive.
d. All the eigenvalues of A are real.

Answer:

1.9 Pick out the correct statements from the following list:
a. A homomorphic image of a UFD (unique factorization domain) is again a UFD.
b. The element $2 \in \mathbb{Z}[\sqrt{-5}]$ is irreducible in $\mathbb{Z}[\sqrt{-5}]$.
c. Units of the ring $\mathbb{Z}[\sqrt{-5}]$ are the units of $\mathbb{Z}$.
d. The element 2 is a prime element in $\mathbb{Z}[\sqrt{-5}]$.

Answer:

1.10 Let p and q be two distinct primes. Pick the correct statements from the following:
a. $\mathbb{Q}(\sqrt{p})$ is isomorphic to $\mathbb{Q}(\sqrt{q})$ as fields.
b. $\mathbb{Q}(\sqrt{p})$ is isomorphic to $\mathbb{Q}(\sqrt{-q})$ as vector spaces over $\mathbb{Q}$.
c. $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = 4$.
d. $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$.

Answer:

Section 2: Analysis

2.1 Let f be a real valued function on $\mathbb{R}$. Consider the functions

$$w_j(x) \;=\; \sup\{|f(u) - f(v)| \;:\; u, v \in [x - \frac{1}{j}, x + \frac{1}{j}]\},$$

where j is a positive integer and $x \in \mathbb{R}$. Define next,

$$A_{j,n} \;=\; \{x \in \mathbb{R} \;:\; w_j(x) < \frac{1}{n}\}, n = 1, 2, \ldots$$

and

$$A_n \;=\; \cup_{j=1}^{\infty} A_{j,n}, n = 1, 2, \ldots$$

Now let

$$C \;=\; \{x \in \mathbb{R} \;:\; f \;\text{ is continuous at }\; x\}.$$

Express C in terms of the sets A_n.

Answer:

2.2 Let f be a continuous real valued function on $\mathbb{R}$ and n, a positive integer. Find

$$\frac{d}{dx}\int_0^x (2x - t)^n f(t)dt.$$

Answer:

2.3 For each $n \geq 1$, let f_n be a monotonic increasing real valued function on $[0, 1]$ such that the sequence of functions $\{f_n\}$ converges pointwise to the function $f \equiv 0$. Pick out the true statements from the following:

a. f_n converges to f uniformly.

b. If the functions f_n are also non-negative, then f_n must be continuous for sufficiently large n.

Answer:

2.4 Let $\mathbb{Q}$ denote the set of all rational numbers in the open interval $]0,1[$. Let $\lambda(U)$ denote the Lebesgue measure of a subset U of $]0,1[$. Pick out the correct statements from the following:
a. $\lambda(U) = 1$ for every open set $U \subset]0,1[$ which contains $\mathbb{Q}$.
b. Given any $\varepsilon > 0$, there exists an open set $U \subset]0,1[$ containing $\mathbb{Q}$ such that $\lambda(U) < \varepsilon$.

Answer:

2.5 A real valued function on an interval $[a,b]$ is said to be a function of bounded variation if there exists $M > 0$, such that for any finite set of points $a = a_0 < a_1 < a_2 < \ldots < a_n = b$, we have $\sum_{i=0}^{n-1} |f(a_i) - f(a_{i+1})| < M$. Which of the following statements are necessarily true ?
a. Any continuous function on $[0,1]$ is of bounded variation.
b. If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then its restriction to the interval $[-n,n]$ is of bounded variation on that interval, for any positive integer n.
c. Any monotone function on $[0,1]$ is of bounded variation.

Answer:

2.6 Let f be a differentiable function of one variable and let g be the function of two variables given by $g(x,y) = f(ax+by)$, where a, b are fixed nonzero numbers. Write down a partial differential equation satisfied by the function g.

Answer:

2.7 The curve $x^3 - y^3 = 1$ is asymptotic to the line $x = y$. Find the point on the curve farthest from the line $x = y$.

Answer:

2.8 Let k be a fixed positive integer. Find R_k, the radius of convergence of the power series $\sum(\frac{n+1}{n})^{n^2} z^{kn}$.

Answer:

2.9 let γ be a closed and continuously differentiable path in the upper half plane

$$\{z \in \mathbb{C} \ : \ z = x + iy, \ x, y \in \mathbb{R}, \ y > 0\}$$

not passing through the point i. Describe the set of all possible values of the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{2i}{z^2 + 1} dz.$$

Answer:

2.10 Let f be a function of three (real) variables having continuous partial derivatives. For each direction vector $h = (h_1, h_2, h_3)$ such that $h_1^2 + h_2^2 + h_3^2 = 1$, let $D_h f(x, y, z)$ be the directional derivative of f along h at (x, y, z). For a point (x_0, y_0, z_0) where the partial derivative $\frac{\partial}{\partial x} f(x_0, y_0, z_0)$ is not zero, maximize $D_h f(x_0, y_0, z_0)$ (as a function of h).

Answer: The maximum value =

Section 3: Topology

3.1 Let f be the function on $\mathbb{R}$ defined by $f(t) = \frac{p+\sqrt{2}}{q+\sqrt{2}} - \frac{p}{q}$ if $t = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and p and q coprime to each other, and $f(t) = 0$ if t is irrational. Answer the following: i) At which irrational numbers t is f is continuous? ii) At which rational numbers t is f continuous?

Answer: i) The set of irrational t where f is continuous:

ii) The set of rational t where f is continuous:

3.2 Let f and g be two continuous functions on $\mathbb{R}$. For any $a \in \mathbb{R}$ we define $J_a(f,g)$ to be the function given by $J_a(f,g)(t) = f(t)$ for all $t \leq a$ and $J_a(f,g)(t) = g(t)$ if $t > a$. For what values of a is $J_a(f,g)$ a continuous function?

Answer: $J_a(f,g)$ is continuous if and only if ....

3.3 Let A and B be two finite subsets of $\mathbb{R}$. Describe a necessary and sufficient condition for the spaces $\mathbb{R}\backslash A$ and $\mathbb{R}\backslash B$ to be homeomorphic.

Answer: $\mathbb{R}\backslash A$ and $\mathbb{R}\backslash B$ are homeomorphic if and only if ....

3.4 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Let D be the closed unit disc in $\mathbb{R}^2$. Is $f(D)$ necessarily and interval in $\mathbb{R}$? If it is an interval, which of the forms $]a,b[$, $[a,b[$, $]a,b]$ and $[a,b]$, with $a, b \in \mathbb{R}$ can it have?

Answer: i) $f(D)$ is necessarily an interval in $\mathbb{R}$/may not be an interval;
ii) Possible form(s) for the interval: ....

3.5 For $v \in \mathbb{R}^2$ and $r > 0$ let $D(v, r)$ denote the closed disc with centre at v and radius r. Let $v = (5, 0) \in \mathbb{R}^2$. For $\alpha > 0$ let X_α be the subset

$$X_\alpha \;=\; D(-v, 3) \cup D(v, 3) \cup \{(x, \alpha x) \;:\; x \in \mathbb{R}\}.$$

Determine the condition on α for X_α to be connected; when it is not connected how many connected components does X_α have?

Answer: i) X_α is connected if and only if ....
ii) When not connected it has ..... connected components.

3.6 Which two of the following spaces are homeomorphic to each other?
i) $X_1 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; xy = 0\}$;
ii) $X_2 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; x + y \geq 0 \text{ and } \; xy = 0\}$;
iii) $X_3 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; xy = 1\}$;
iv) $X_4 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; x + y \geq 0, \text{ and } \; xy = 1\}$.

Answer The sets ...... and ...... are homeomorphic.

3.7 Which of the following spaces are compact?
i) $X_1 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; |x| + |y| < 10^{-100}\}$;
ii) $X_2 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; |x| + |y| \leq 10^{100}\}$;
iii) $X_3 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; 1 \leq x^2 + y^2 \leq 2\}$;
iv) $X_4 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 = 1 \text{ and } \; xy \neq 0\}$.

Answer: Compact subsets from the above are ......

3.8 Which of the following spaces are locally compact?
i) $X_1 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; x, y \text{ odd integers}\}$;
ii) $X_2 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; x^2 + 103xy + 7y^2 > 5\}$;
iii) $X_3 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; 0 \leq x < 1, 0 < y \leq 1\}$;
iv) $X_4 \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; x, y \text{ irrational}\}$.

Answer: Locally compact spaces from the above are ......

3.9 Which of the following metric spaces (X_i, d_i), $1 \leq i \leq 4$, are complete?
i) $X_1 =]0, \pi/2[\subset \mathbb{R}$, d_1 defined by $d_1(x, y) = |\tan x - \tan y|$ for all $x, y \in X_1$.
ii) $X_2 = [0, 1] \subset \mathbb{R}$, d_2 defined by $d_2(x, y) = \frac{|x-y|}{1+|x-y|}$ for all $x, y \in X_2$.
iii) $X_3 = \mathbb{Q}$, and d_3 defined by $d_3(x, y) = 1$ for all $x, y \in X_3,\ x \neq y$.
iv) $X_4 = \mathbb{R}$, d_4 defined by $d_4(x, y) = |e^x - e^y|$ for all $x, y \in X_4$.

Answer: Complete metric spaces from the above are ......

3.10 On which of the following spaces is every continuous (real-valued) function bounded?
i) $X_1 =]0, 1[$;
ii) $X_2 = [0, 1]$;
iii) $X_3 = [0, 1[$;
iv) $X_4 = \{t \in [0, 1] \ : \ t \text{ irrational}\}$.

Answer: Every continuous function on ....
is bounded (enter all X_i with i between 1 and 4 for which the statement holds).

Section 4: Applied Mathematics

4.1 Let $\Gamma(s)$ stand for the usual Gamma function. Given that $\Gamma(1/2) = \sqrt{\pi}$, evaluate $\Gamma(5/2)$.

Answer:

4.2 Let

$$S \;=\; \{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = 1,\; z > 0\}.$$

Let

$$C \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 = 1\}.$$

Let τ be the unit tangent vector to C in the xy-plane pointing left as we move clockwise along C. Let $\varphi(x, y, z) = x^2 + y^3 + z^4$. Evaluate:

$$\int_C \nabla\varphi.\tau \; ds.$$

Answer:

4.3 Let $a > 0$ and let

$$S \;=\; \{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = a^2\}.$$

Evaluate:

$$\int\int_S (x^4 + y^4 + z^4)\; dS.$$

Answer:

4.4 Let $f(x) = x^2 - 5$ for $x \in \mathbb{R}$. Let $x_0 = 1$. If $\{x_n\}$ denotes the sequence of iterates defined by the Newton-Raphson method to approximate a solution of $f(x) = 0$, find x_1.

Answer:

4.5 Let A be a 2×2 matrix with real entries. Consider the linear sysytem of ordinary differential equations given in vector notation as:

$$\frac{d\mathbf{x}}{dt}(t) \;=\; A\mathbf{x}(t)$$

where

$$\mathbf{x}(t) \;=\; \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

Pick out the cases from the following when we have $\lim_{t\to\infty} u(t) = 0$ and $\lim_{t\to\infty} v(t) = 0$:

a.

$$A \;=\; \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

b.

$$A \;=\; \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}.$$

c.

$$A \;=\; \begin{pmatrix} 1 & -6 \\ 1 & -4 \end{pmatrix}.$$

d.

$$A \;=\; \begin{pmatrix} -1 & -6 \\ 1 & 4 \end{pmatrix}.$$

Answer:

4.6 Let $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denote the Laplace operator. Let

$$\Omega \;=\; \{(x,y) \in \mathbb{R}^2 \;:\; x^2 + y^2 < 1\}.$$

Let $\partial\Omega$ denote the boundary of the domain Ω. Consider the following boundary value problem:

$$\begin{aligned} \Delta u &= c \text{ in } \Omega \\ \tfrac{\partial u}{\partial \nu} &= 1 \text{ on } \partial\Omega \end{aligned}$$

where c is a real constant and $\partial u/\partial\nu$ denotes the outward normal derivative of u on $\partial\Omega$. For what values of c does the above problem admit a solution?

Answer:

4.7 Consider the Tricomi equation:

$$\frac{\partial^2 u}{\partial y^2} - y\frac{\partial^2 u}{\partial x^2} \ = \ 0.$$

Describe the region in the xy-plane where this equation is elliptic.

Answer:

4.8 Evaluate:

$$\int\int_{\mathbb{R}^2} e^{-(3x+2y)^2-(4x+y)^2}\,dxdy.$$

Answer:

4.9 Let J_p denote the Bessel function of the first kind, of order p and let $\{P_n\}$ denote the sequence of Legendre polynomials defined on the interval $[-1.1]$. Pick out the true statements from the following:
a. $\frac{d}{dx}J_o(x) = -J_1(x)$.
b. Between any two positive zeroes of J_0, there exists a zero of J_1.
c. $P_{n+1}(x)$ can be written as a linear combination of $P_n(x)$ and $P_{n-1}(x)$.
d. $P_{n+1}(x)$ can be written as a linear combination of $xP_n(x)$ and $P_{n-1}(x)$.

Answer:

4.10 Consider the linear programming problem: Maximize $z = 2x_1+3x_2+x_3$ such that

$$\begin{aligned} 4x_1 + 3x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 5x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Write down the objective function of the dual problem.

Answer:

Section 5: Miscellaneous

5.1 A unimodular matrix is a matrix with integer entries and having determinant 1 or -1. If m and n are positive integers, write down a necessary and sufficient condition such that there exists a unimodular matrix of order 2 whose first row is the vector (m, n).

Answer:

5.2 For any integer n define $k(n) = \frac{n^7}{7} + \frac{n^3}{3} + \frac{11n}{21} + 1$ and

$$f(n) = \begin{cases} 0 & \text{if } k(n) \text{ an integer,} \\ \frac{1}{n^2} & \text{if } k(n) \text{ is not an integer.} \end{cases}$$

Find $\displaystyle\sum_{n=-\infty}^{\infty} f(n)$.

Answer:

5.3 Let $n \geq 2$. Evaluate:

$$\sum_{k=2}^{n} \frac{n!}{(n-k)!(k-2)!}.$$

Answer:

5.4 A fair coin is tossed ten times. What is the probability that we can observe a string of eight heads, in succession, at some time?

Answer:

5.5 Evaluate the product $\displaystyle\prod_{n=2}^{\infty} \left(1 + \frac{1}{n^2} + \frac{1}{n^4} + \frac{1}{n^6} + ...\right)$.

Answer:

5.6 Find all solutions of the equation

$$(x^2 + y^2 + z^2 - 1)^2 + (x + y + z - 3)^2 = 0.$$

Answer:

5.7 For any real number x, let $f(x)$ denote the distance of x from the nearest integer. Let $I(k) = [k\pi, k\pi + 1]$. Find $f(I(k))$ for all integers k.

Answer:

5.8 Let K be a finite field. Can you always find a non-constant polynomial over K which has no root in K ? If yes, give one such polynomial.

Answer: No, there is no such polynomial/ Yes, and one such polynomial is given by:

5.9 Evaluate:

$$\sum_{k=1}^{\infty} \frac{k^2}{k!}.$$

Answer:

5.10 Pick out the countable sets from the following:
a. $\{\alpha \in \mathbb{R} \ : \ \alpha$ is a root of a polynomial with integer coefficients$\}$.
b. The complement in $\mathbb{R}$ of the set described in statement (a) above.
c. The set of all points in the plane whose coordinates are rational.
d. Any subset of $\mathbb{R}$ whose Lebesgue measure is zero.

Answer: